

Error exponents in hypothesis testing for correlated states on a spin chain

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Abstract

We study various error exponents in a binary hypothesis testing problem and extend recent results on the quantum Chernoff and Hoeffding bounds for product states to a setting when both the null-hypothesis and the counter-hypothesis can be correlated states on a spin chain. Our results apply to states satisfying a certain factorization property; typical examples are the global Gibbs states of translation-invariant finite-range interactions as well as certain finitely correlated states.

Keywords: Hypothesis testing, Chernoff bound, Hoeffding bound, Stein's lemma, spin chains.

1 Introduction

We study the asymptotics of the error probabilities in a binary hypothesis testing problem for quantum systems. In a rather general setting (used generally in the information-spectrum approach [7, 19]), one can consider a sequence of finite-level quantum systems with (finite-dimensional) Hilbert spaces $\vec{\mathcal{H}} = \{\mathcal{H}_n\}_{n=1}^{\infty}$. Assume that we know a priori that the n th system is in state ρ_n (null-hypothesis H_0) or in state σ_n (counter-hypothesis H_1). The hypothesis testing problem for the n th system is to decide between the above two options, based on the outcome of a binary measurement on the system.

A measurement in our setting means a binary positive operator valued measure $\{T_n, I_n - T_n\}$ where $0 \leq T_n \leq I_n$ corresponds to outcome 0 and $I_n - T_n$ to outcome 1. If the outcome of the measurement is 0 (resp. 1) then hypothesis H_0 (resp. H_1) is accepted. Obviously we can identify the measurement with the single operator T_n . An erroneous decision is

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made if H_1 (H_0) is accepted when the true state of the system is ρ_n (σ_n); the probabilities of these events are the error probabilities of the first (second) kinds, given by

$$\alpha_n(T_n) := \rho_n(I_n - T_n) = \text{Tr } \hat{\rho}_n(I_n - T_n) \quad \text{and} \quad \beta_n(T_n) := \sigma_n(T_n) = \text{Tr } \hat{\sigma}_n T_n,$$

respectively. (Here $\hat{\omega}$ denotes the density operator of a state ω , given by $\omega(A) = \text{Tr } \hat{\omega} A$, $A \in \mathcal{B}(\mathcal{H}_n)$.)

Apart from the trivial case when $\text{supp } \hat{\rho}_n \perp \text{supp } \hat{\sigma}_n$, one cannot find a measurement making both error probabilities to vanish; in general, there is a tradeoff between the two. In the general cases of interest the error probabilities are expected to tend to zero asymptotically (typically with an exponential speed) when the measurements T_n are chosen in an optimal way. In the asymmetric setting of *Stein's lemma* [5, 12] the exponential decay of the β_n 's is studied either under the constraint that the α_n 's tend to 0, or that the α_n 's stay under a constant bound. As it was shown in [12] and [21], in the i.i.d. case (i.e. when $\mathcal{H}_n = \mathcal{H}_1^{\otimes n}$, $\rho_n = \rho_1^{\otimes n}$ and $\sigma_n = \sigma_1^{\otimes n}$) the optimal exponential decay rate is given by $-S(\rho_1 \parallel \sigma_1)$, the negative relative entropy of ρ_1 and σ_1 , thus giving an operational interpretation to relative entropy. This result was later extended to cases when the sequences $\vec{\rho} := \{\rho_n\}_{n=1}^\infty$ and $\vec{\sigma} := \{\sigma_n\}_{n=1}^\infty$ consist of restrictions of an ergodic state ρ and a shift-invariant product state σ on a spin chain [13, 3]. In the symmetric setting of the *Chernoff bound* [1, 2, 20] the exponential decay of the average of the two error probabilities is of interest. As it was shown in [1] and [20], the best exponential decay rate in the i.i.d. case is given by $-C(\rho_1, \sigma_1)$, with

$$C(\rho_1, \sigma_1) := -\min_{0 \leq s \leq 1} \psi(s), \quad \psi(s) := \log \text{Tr } \hat{\rho}_1^s \hat{\sigma}_1^{1-s}. \quad (1)$$

The above result shows that the quantity C plays a similar role in symmetric hypothesis testing as the relative entropy does in the asymmetric case.

When an exponential bound is given on the decay of the α_n 's, our interest lies in the following quantities [22, 19]:

$$\underline{B}(r|\vec{\rho}||\vec{\sigma}) := \inf_{\{T_n\}} \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \mid \limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(T_n) < -r \right\}, \quad (2)$$

$$\overline{B}(r|\vec{\rho}||\vec{\sigma}) := \inf_{\{T_n\}} \left\{ \limsup_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \mid \limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(T_n) < -r \right\}, \quad (3)$$

$$B(r|\vec{\rho}||\vec{\sigma}) := \inf_{\{T_n\}} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \mid \limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(T_n) < -r \right\}. \quad (4)$$

Based on the techniques developed in [1] and [20], the identity

$$\overline{B}(r|\vec{\rho}||\vec{\sigma}) = B(r|\vec{\rho}||\vec{\sigma}) = -b(r), \quad b(r) := \max_{0 \leq s < 1} \frac{-sr - \psi(s)}{1 - s} \quad (5)$$

was proven for $0 < r \leq S(\rho_1 \parallel \sigma_1)$ in the i.i.d. setting in [8] (where $\overline{B}(r|\vec{\rho}||\vec{\sigma}) \leq -b(r)$ was shown) and [17] (where the inequality $B(r|\vec{\rho}||\vec{\sigma}) \geq -b(r)$ was provided), thus establishing the theorem for the quantum *Hoeffding bound*.

In this paper we will mainly consider the situation when ρ_n and σ_n are the n -site restrictions of states ρ and σ on an infinite spin chain, satisfying a certain factorization

property. Typical examples of such states are the global Gibbs states of translation-invariant finite-range interactions [10] and certain finitely correlated states [6, 10]. Our main result is that (5) holds for such states when ψ in (1) is replaced with

$$\psi(s) := \lim_n \frac{1}{n} \log \text{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s}, \quad s \in [0, 1].$$

As a side-result, we recover the quantum Chernoff bound (already proven in [10]) and a Stein-type upper bound for states of the above type.

2 Preliminaries and upper bounds

2.1 Error exponents: upper bounds

Let A and B be nonnegative operators on a finite-dimensional Hilbert space \mathcal{H} . It is easy to see that

$$\min_{0 \leq T \leq I} \{ \text{Tr} A(I - T) + \text{Tr} BT \} = \frac{1}{2} \text{Tr}(A + B) - \frac{1}{2} \text{Tr} |A - B|, \quad (6)$$

and the minimum is attained at $\{A - B > 0\}$, the spectral projection of $A - B$ corresponding to the positive part of the spectrum. The following was shown in [1]:

Lemma 2.1. Let A and B be nonnegative operators on a finite-dimensional Hilbert space \mathcal{H} . Then

$$\frac{1}{2} \text{Tr}(A + B) - \frac{1}{2} \text{Tr} |A - B| \leq \text{Tr} A^s B^{1-s}, \quad 0 \leq s \leq 1. \quad (7)$$

All along the paper we use the convention $0^s := 0$, $s \in \mathbb{R}$; in particular, A^0 and B^0 are defined to be the support projections of A and B , respectively. With this convention $s \mapsto \text{Tr} A^s B^{1-s}$ is a continuous function on \mathbb{R} . In Appendix A we mention another representation of the quantity (6) given in [25].

Consider now the hypothesis testing problem described in the Introduction, and assume that we know a priori that the n th system is in the state ρ_n with probability $\pi_n \in (0, 1)$ or in the state σ_n with probability $1 - \pi_n$. Then the Bayesian probability of an erroneous decision based on the test T_n is

$$P_{T_n}(\rho_n : \sigma_n | \pi_n) := \pi_n \alpha_n(T_n) + (1 - \pi_n) \beta_n(T_n),$$

and by applying Lemma 2.1 to $A := \pi_n \hat{\rho}_n$ and $B := (1 - \pi_n) \hat{\sigma}_n$ we get

$$P_{\min}(\rho_n : \sigma_n | \pi_n) := \min_{0 \leq T_n \leq I_n} \{ P_{T_n}(\rho_n : \sigma_n | \pi_n) \} \leq \pi_n^s (1 - \pi_n)^{1-s} \text{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s}. \quad (8)$$

Note that the optimal test is of the form $\{\pi_n \hat{\rho}_n - (1 - \pi_n) \hat{\sigma}_n > 0\}$. Let

$$\psi_n(s) := \frac{1}{n} \log \text{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s}, \quad s \in \mathbb{R}. \quad (9)$$

It is easy to see that ψ_n is a convex function on \mathbb{R} for all $n \in \mathbb{N}$. Next, let

$$\psi(s) := \limsup_{n \rightarrow \infty} \psi_n(s), \quad s \in [0, 1],$$

and

$$\varphi(a) := \sup_{0 \leq s \leq 1} \{as - \psi(s)\}, \quad a \in \mathbb{R}, \quad (10)$$

be its *conjugate* (or *polar*) function. If $\pi_n = \pi$ is independent of n then (8) implies

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{\min}(\rho_n : \sigma_n | \pi) \leq \inf_{0 \leq s \leq 1} \psi(s) = -\varphi(0), \quad (11)$$

as it was pointed out in the i.i.d. case in [1].

After this preparation, we prove the following:

Lemma 2.2. Let $\hat{\rho}_n$ and $\hat{\sigma}_n$ be density operators on a Hilbert space \mathcal{H}_n for each $n \in \mathbb{N}$. Then for any $a \in \mathbb{R}$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \min_{0 \leq T_n \leq I_n} \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} \leq -\varphi(a). \quad (12)$$

Moreover, for $S_{n,a} := \{e^{-na} \hat{\rho}_n - \hat{\sigma}_n > 0\}$ we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(S_{n,a}) &\leq -\{\varphi(a) - a\}, \\ \limsup_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(S_{n,a}) &\leq -\varphi(a). \end{aligned}$$

Proof. Consider formula (8) with $\pi_n := \frac{e^{-na}}{1+e^{-na}}$ for a fixed $a \in \mathbb{R}$. The optimal test is then $\{\pi_n \hat{\rho}_n - (1 - \pi_n) \hat{\sigma}_n > 0\} = S_{n,a}$, and by multiplying (8) by $1 + e^{-na}$ we get

$$\min_{0 \leq T_n \leq I_n} \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} = e^{-na} \alpha_n(S_{n,a}) + \beta_n(S_{n,a}) \leq e^{-nas} \text{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s},$$

which implies

$$\alpha_n(S_{n,a}) \leq e^{na} e^{-nas} \text{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s}, \quad \beta_n(S_{n,a}) \leq e^{-nas} \text{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s}$$

for any $s \in [0, 1]$. Thus the statement follows. \square

Consider now an asymmetric hypothesis testing problem with an exponential bound on the decay of the α_n 's. The relevant error exponents in this case are given in (2), (3) and (4). Obviously for any fixed $r \in \mathbb{R}$

$$\underline{B}(r|\vec{\rho}||\vec{\sigma}) \leq \overline{B}(r|\vec{\rho}||\vec{\sigma}) \leq B(r|\vec{\rho}||\vec{\sigma}), \quad (13)$$

and all the above quantities are monotonically increasing functions of r . Note that for $r < 0$ the choice $T_n := I_n$ yields $\underline{B}(r|\vec{\rho}||\vec{\sigma}) = \overline{B}(r|\vec{\rho}||\vec{\sigma}) = B(r|\vec{\rho}||\vec{\sigma}) = -\infty$, hence the above quantities are only interesting for $r \geq 0$.

Lemma 2.2 yields the following corollary, that can be considered as the direct part of the theorem for the quantum Hoeffding bound:

Corollary 2.3. In the above setting

$$\overline{B}(r|\vec{\rho}||\vec{\sigma}) \leq - \sup_{a: \varphi(a) - a > r} \varphi(a).$$

The converse part of (11), inequality

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P_{\min}(\rho_n : \sigma_n | \pi) \geq \inf_{0 \leq s \leq 1} \psi(s) = -\varphi(0) \quad (14)$$

was shown in the i.i.d. setting in [20]. Inequalities (11) and (14) together give the theorem for the quantum *Chernoff bound* in the i.i.d. case. The main idea in proving the lower bound is to relate the problem to the classical hypothesis testing problem of a certain pair of classical probability measures associated to the original pair of quantum states. The same method was used to prove the lower bound in the theorem for the quantum Hoeffding bound in [17]. In Section 3.1 we follow (a slight modification of) this method to show that the converse part of inequality (12) in Lemma 2.2 holds (in the above general setting) if the functions ψ_n converge to a differentiable function on \mathbb{R} . Apart from yielding the lower bound in the i.i.d. setting as a special case, there are examples for correlated states on a spin chain for which this criterion can be verified (see Example B.1). In general, however, differentiability seems to be rather difficult to prove, therefore we follow a different approach in Section 3.2 to obtain the converse part for a certain class of states on a spin chain, which we introduce in Section 2.2.

2.2 Spin chains and factorization property

Let \mathcal{H} be a finite-dimensional Hilbert space and $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^* -subalgebra. For all $k, l \in \mathbb{Z}$, $k \leq l$, the finite-size algebra $\mathcal{C}_{[k,l]} := \otimes_{k \leq i \leq l} \mathcal{A}$ is naturally embedded into all $\mathcal{C}_{[m,n]}$ with $m \leq k, n \geq l$, hence one can define $\mathcal{C}_{\text{loc}} := \bigcup_{k,l \in \mathbb{N}} \mathcal{C}_{[k,l]}$, which is a pre- C^* -algebra with unit $\mathbb{1}$. The *spin chain* \mathcal{C} with one-site algebra \mathcal{A} is the uniform closure of \mathcal{C}_{loc} . It is natural to consider \mathcal{C} as the infinite tensor power of \mathcal{A} , hence the notation $\mathcal{C} = \otimes_{k \in \mathbb{Z}} \mathcal{A}$ is also used. The right shift automorphism γ is the unique extension of the maps $\gamma_{kl} : \mathcal{C}_{[k,l]} \rightarrow \mathcal{C}_{[k,l+1]}$, $X \mapsto \mathbb{1}_{\mathcal{A}} \otimes X$.

States on the spin chain are positive linear functionals on \mathcal{C} that take the value 1 on $\mathbb{1}$. A state ω is *translation-invariant* if $\omega \circ \gamma = \omega$ holds. A translation-invariant state ω is uniquely determined by $\vec{\omega} := \{\omega_n\}_{n=1}^{\infty}$, where ω_n is its restriction onto $\mathcal{C}_{[1,n]}$.

Definition 2.4. A translation-invariant state ω on the spin chain satisfies upper/lower factorization properties if there exists a positive constant $\eta \in \mathbb{R}$ such that

$$\omega \leq \eta \omega_{(-\infty,0]} \otimes \omega_{[1,+\infty)} \quad (\text{upper factorization}), \quad (15)$$

$$\omega \geq \eta^{-1} \omega_{(-\infty,0]} \otimes \omega_{[1,+\infty)} \quad (\text{lower factorization}). \quad (16)$$

For a fixed $m \in \mathbb{N}$ any number $n \in \mathbb{N}$ can be written in the form $n = km + r$ with $k, r \in \mathbb{N}$, $1 \leq r \leq m$, and consecutive applications of the above inequalities give

$$\omega_{[1,n]} \leq \eta^k \omega_{[1,m]}^{\otimes k} \otimes \omega_{[1,r]} \quad (\text{upper factorization}), \quad (17)$$

$$\omega_{[1,n]} \geq \eta^{-k} \omega_{[1,m]}^{\otimes k} \otimes \omega_{[1,r]} \quad (\text{lower factorization}). \quad (18)$$

On the other hand, it is easily seen by taking $n = 2m$ for an arbitrarily large m that inequalities (17) and (18) imply (15) and (16), respectively. We will use the notation $\vec{\rho} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$ if ρ_n , $n \in \mathbb{N}$, are the n -site restrictions of a shift-invariant state ρ on \mathcal{C} that satisfies the factorization properties above.

Obviously, a product state $\omega := \omega_1^{\otimes \infty}$ satisfies both upper and lower factorization properties. As it was shown in [10], finitely correlated states [6] satisfy upper factorization property, and in some special cases (e.g. locally faithful Markov states) also lower factorization property [10, 11]. Another important class of states that satisfy both upper and lower factorization properties is that of the global Gibbs states of translation-invariant finite-range interactions. This result was also shown in [10], based on the perturbation bounds developed in [16].

Now let $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$. Without loss of generality we can assume that they have the same factorization constant η . If $\text{supp } \hat{\rho}_m \perp \text{supp } \hat{\sigma}_m$ for some m then the upper factorization property (15) yields that $\text{supp } \hat{\rho}_n \perp \text{supp } \hat{\sigma}_n$ for all $n \in \mathbb{N}$. Since in this case the hypothesis testing problem is trivial, we will always assume that $\text{supp } \hat{\rho}_n$ and $\text{supp } \hat{\sigma}_n$ are not orthogonal to each other for any n , as far as the case $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$ is concerned.

Let ψ_n , $n \in \mathbb{N}$, be as given in (9). The following lemma was shown in [10]; for readers' convenience and because we need an intermediate formula for later purposes, we give a detailed proof here.

Lemma 2.5. The limit $\psi(s) := \lim_{n \rightarrow \infty} \psi_n(s)$ exists for all $s \in [0, 1]$, and ψ is a convex and continuous function on $[0, 1]$.

Proof. Let $m \in \mathbb{N}$ be fixed and $n = km + r$, $1 \leq r \leq m$. Upper factorization property together with the operator monotonicity of the function $x \mapsto x^s$, $0 \leq s \leq 1$, implies

$$\text{Tr } \hat{\rho}_n^s \hat{\sigma}_n^{1-s} \leq M \eta^k (\text{Tr } \hat{\rho}_m^s \hat{\sigma}_m^{1-s})^k,$$

where $M = \max\{\text{Tr } \hat{\rho}_r^s \hat{\sigma}_r^{1-s} : 1 \leq r \leq m, 0 \leq s \leq 1\}$, and hence

$$\psi_n(s) \leq \frac{1}{n} \log M + \frac{k}{n} \log \eta + \frac{km}{n} \psi_m(s).$$

Taking the lim sup in n , we obtain

$$\limsup_{n \rightarrow \infty} \psi_n(s) \leq \psi_m(s) + \frac{1}{m} \log \eta.$$

Taking the lim inf in m then gives the existence of the limit. Being the pointwise limit of convex functions, ψ is convex (and hence continuous in $(0, 1)$).

In the same way as above, lower factorization property implies

$$\psi_m(s) - \frac{1}{m} \log \eta \leq \liminf_{n \rightarrow \infty} \psi_n(s)$$

and we obtain the bound

$$\psi_m(s) - \frac{1}{m} \log \eta \leq \psi(s) \leq \psi_m(s) + \frac{1}{m} \log \eta \quad (19)$$

for every $m \in \mathbb{N}$. This shows that ψ is the uniform limit of the ψ_n 's, and hence the continuity of ψ follows. \square

3 A Chernoff-type theorem

In this section we complement inequality (12) of Lemma 2.2. Our main interest is in the situation when $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$; we treat this case in Section 3.2. The main idea in this case is to use the lower factorization property to reduce the problem to the i.i.d. setting.

In Section 3.1 we prove the converse inequality of (12) under the assumption that the functions ψ_n defined in (9) converge to a differentiable function ψ on \mathbb{R} . Even though this condition may seem to be rather abstract and difficult to verify in general, this approach has at least two merits. First, the converse inequality for the i.i.d. situation (needed in Section 3.2) follows as a special case. Second, it provides an extension from the i.i.d. situation that can be different from requiring the lower factorization property to hold, as we point out in Remark B.2.

3.1 Lower bound under differentiability

Let ψ_n , $n \in \mathbb{N}$, be as given in (9) and assume that

- (a1) the limit $\psi(s) := \lim_n \psi_n(s)$ exists as a real number for all $s \in \mathbb{R}$;
- (a2) ψ is a differentiable function on \mathbb{R} .

Note that assumption (a1) implies that $\text{supp } \hat{\rho}_n$ cannot be orthogonal to $\text{supp } \hat{\sigma}_n$, except for finitely many n 's. Since all the ψ_n 's are convex on \mathbb{R} , ψ is a convex function on \mathbb{R} as well. Let $\tilde{\psi}(s) := \psi(1-s)$, $s \in \mathbb{R}$; then

$$\psi^*(x) := \sup_{s \in \mathbb{R}} \{sx - \psi(s)\} \quad \text{and} \quad (\tilde{\psi})^*(x) := \sup_{s \in \mathbb{R}} \{sx - \tilde{\psi}(s)\} = x + \psi^*(-x)$$

are convex functions on \mathbb{R} with values in $(-\infty, +\infty]$ (usually referred to as the *Legendre-Fenchel transforms* of ψ and $\tilde{\psi}$). Let $\varphi(a) := \sup_{0 \leq s \leq 1} \{as - \psi(s)\}$, $a \in \mathbb{R}$, as given in (10); then $\varphi(a) \leq \psi^*(a)$ and $\varphi(a) = \psi^*(a)$ if and only if $\psi'(0) \leq a \leq \psi'(1)$.

We will use (a slight modification of) the method of [20] and [17] together with the Gärtner-Ellis theorem (see e.g. [5, Section 2.3]) to show the following:

Theorem 3.1. Under the above assumptions

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \min_{0 \leq T_n \leq I_n} \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} = -\varphi(a) \quad (20)$$

for any $a \in \mathbb{R}$ with $a \neq \psi'(0), \psi'(1)$. Moreover, if ψ^* is continuous at $\psi'(0)$ and at $\psi'(1)$ then (20) holds for all $a \in \mathbb{R}$.

Proof. Thanks to Lemma 2.2 it suffices to prove that for any sequence of tests $\{T_n\}$ we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} \geq -\varphi(a). \quad (21)$$

Let

$$\hat{\rho}_n = \sum_{i \in I_n} \lambda_i P_i, \quad \hat{\sigma}_n = \sum_{j \in J_n} \eta_j Q_j \quad (22)$$

be a decomposition of the densities $\hat{\rho}_n$ and $\hat{\sigma}_n$, where P_i, Q_j are projections and $\lambda_i, \eta_j > 0$ for all $i \in I_n, j \in J_n$. Define the classical discrete positive measures on $I_n \times J_n$ by

$$p_n(i, j) := \lambda_i \operatorname{Tr} P_i Q_j, \quad q_n(i, j) := \eta_j \operatorname{Tr} P_i Q_j.$$

Note that $p_n(I_n \times J_n) \leq 1$ and $q_n(I_n \times J_n) \leq 1$, and $\operatorname{supp} p_n = \operatorname{supp} q_n$ holds for all n . Moreover, it is easy to see that $\operatorname{Tr} \hat{\rho}_n^s \hat{\sigma}_n^{1-s} = \sum_{(i,j) \in I_n \times J_n} p_n(i, j)^s q_n(i, j)^{1-s}$ and assumption (a1) implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log p_n(I_n \times J_n) = \psi(0) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log q_n(I_n \times J_n) = \psi(1). \quad (23)$$

Let $S_{n,a} := \{e^{-na} \hat{\rho}_n - \hat{\sigma}_n > 0\}$. Then

$$\begin{aligned} e_n(a) &:= \min_{0 \leq T_n \leq I_n} \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} = e^{-na} \operatorname{Tr} \hat{\rho}_n(I - S_{n,a}) + \operatorname{Tr} \hat{\sigma}_n S_{n,a} \\ &= e^{-na} \sum_{i \in I_n} \lambda_i \operatorname{Tr}(I - S_{n,a}) P_i(I - S_{n,a}) + \sum_{j \in J_n} \eta_j \operatorname{Tr} S_{n,a} Q_j S_{n,a} \\ &\geq e^{-na} \sum_{(i,j) \in I_n \times J_n} \lambda_i \operatorname{Tr}(I - S_{n,a}) P_i(I - S_{n,a}) Q_j + \sum_{(i,j) \in I_n \times J_n} \eta_j \operatorname{Tr} S_{n,a} Q_j S_{n,a} P_i \\ &\geq \sum_{(i,j) \in I_n \times J_n} \min\{e^{-na} \lambda_i, \eta_j\} \operatorname{Tr} [(I - S_{n,a}) P_i(I - S_{n,a}) + S_{n,a} P_i S_{n,a}] Q_j. \end{aligned}$$

Now by [9, Lemma 9] we have

$$\frac{1}{2}(I - S_{n,a}) P_i(I - S_{n,a}) + \frac{1}{2} S_{n,a} P_i S_{n,a} \geq \frac{1}{4} P_i, \quad i \in I_n.$$

(This can also be seen from the operator convexity of the function $f_A : X \mapsto X^* A X$ for a positive semidefinite A , which fact can be verified by a straightforward computation [22, Lemma 5].) As a consequence,

$$\begin{aligned} 2e_n(a) &\geq \sum_{(i,j) \in I_n \times J_n} \min\{e^{-na} \lambda_i, \eta_j\} \operatorname{Tr} P_i Q_j = \sum_{(i,j) \in I_n \times J_n} \min\{e^{-na} p_n(i, j), q_n(i, j)\} \\ &= e^{-na} p_n(\{e^{-na} p_n(i, j) \leq q_n(i, j)\}) + q_n(\{e^{-na} p_n(i, j) > q_n(i, j)\}) \\ &= e^{-na} p_n(\{X_n \geq -a\}) + q_n(\{Y_n > a\}), \end{aligned}$$

where

$$X_n(i, j) := \frac{1}{n} \log \frac{q_n(i, j)}{p_n(i, j)}, \quad Y_n(i, j) := \frac{1}{n} \log \frac{p_n(i, j)}{q_n(i, j)}$$

are random variables with corresponding distribution measures $\mu_n^{(1)} := p_n \circ X_n^{-1}$ and $\mu_n^{(2)} := q_n \circ Y_n^{-1}$. A straightforward computation shows that

$$\log \int e^{nsx} d\mu_n^{(1)} = n\psi_n(1-s), \quad \log \int e^{nsx} d\mu_n^{(2)} = n\psi_n(s).$$

Under our assumptions (a1) and (a2), the Gärtner-Ellis theorem yields that

$$\begin{aligned}\liminf_n \frac{1}{n} \log p_n(\{X_n \geq -a\}) &\geq -\inf_{x > -a} (\tilde{\psi})^*(x), \\ \liminf_n \frac{1}{n} \log q_n(\{Y_n > a\}) &\geq -\inf_{x > a} \psi^*(x),\end{aligned}$$

and therefore

$$\liminf_n \frac{1}{n} \log e_n(a) \geq -m(a) := -\min \left\{ \inf_{x > a} \psi^*(x), a + \inf_{x > -a} (\tilde{\psi})^*(x) \right\}.$$

We remark that the Gärtner-Ellis theorem is usually stated for probability measures while our measures p_n and q_n are in general subnormalized. However, this case follows immediately from the standard version due to the existence of the limits in (23).

Since $\inf_{x \in \mathbb{R}} \psi^*(x) = \psi^*(\psi'(0)) = -\psi(0)$, we get $\inf_{x > a} \psi^*(x) = -\psi(0)$ for $a < \psi'(0)$, and in this case also $\varphi(a) = -\psi(0)$, hence $m(a) \leq \varphi(a)$. Moreover, the same holds for $a = \psi'(0)$ if it is a continuity point of ψ^* . Similarly, $\inf_{x \in \mathbb{R}} (\tilde{\psi})^*(x) = (\tilde{\psi})^*(\tilde{\psi}'(0)) = -\tilde{\psi}(0) = -\psi(1)$ implies $\inf_{x > -a} (\tilde{\psi})^*(x) = -\psi(1)$ for $-a < \tilde{\psi}'(0) = -\psi'(1)$, hence $m(a) \leq a - \psi(1) = \varphi(a)$ when $a > \psi'(1)$. Again, continuity of $(\tilde{\psi})^*$ at $\tilde{\psi}'(0)$ (i.e. continuity of ψ^* at $\psi'(1)$) yields $m(a) \leq \varphi(a)$ for $a = \psi'(1)$. The proof is finished by noting that for $\psi'(0) < a < \psi'(1)$ we have $\inf_{x > a} \psi^*(x) = \psi^*(a) = \varphi(a)$. \square

In the proof of Theorem 3.4 we will need that (20) holds for every $a \in \mathbb{R}$ in the special case when both ρ and σ are shift-invariant product states on a spin chain. To prove this, we first give the following lemma, which may be interesting by itself:

Lemma 3.2. Let a and b positive elements in a C^* -algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, where \mathcal{H} is a finite-dimensional Hilbert space. Let $a = \sum_{i \in I} \lambda_i P_i$ and $b = \sum_{j \in J} \eta_j Q_j$ be their spectral decompositions (with all $\lambda_i, \eta_j > 0$), and let $\psi(s) := \log \text{Tr } a^s b^{1-s}$, $s \in \mathbb{R}$. Then the following are equivalent:

- (i) ψ is an affine function on \mathbb{R} ;
- (ii) there exists an $s \in \mathbb{R}$ such that $\psi''(s) = 0$;
- (iii) there exist $i_1, \dots, i_m \in I$, $j_1, \dots, j_m \in J$ and a $\delta > 0$ such that

$$P_{i_k} \vee Q_{j_k} \perp P_{i_l} \vee Q_{j_l}, \quad k \neq l,$$

$$P_{i_k} \wedge Q_{j_k} \neq 0 \quad \text{and} \quad \lambda_{i_k} = \delta \eta_{j_k}, \quad k = 1, \dots, m,$$

and that $\sum_{i \in I \setminus \{i_1, \dots, i_m\}} P_i$, $\sum_{j \in J \setminus \{j_1, \dots, j_m\}} Q_j$ and $\sum_{k=1}^m P_{i_k} \vee Q_{j_k}$ are mutually orthogonal projections.

Moreover, if \mathcal{A} is isomorphic to the function algebra on a finite set \mathcal{X} then the above are also equivalent to

- (iv) There exists a $\delta > 0$ such that $a(x) = \delta b(x)$ for every $x \in \mathcal{X}$ with $a(x)b(x) \neq 0$.

Proof. (i) \Rightarrow (ii) is obvious, and (iii) \Rightarrow (i) is easy to check.

To see (ii) \Rightarrow (iii), define a function f and probability distributions p_s , $s \in \mathbb{R}$, on $I \times J$ by

$$f(i, j) := \log \lambda_i - \log \eta_j, \quad p_s(i, j) := \frac{\lambda_i^s \eta_j^{1-s} \operatorname{Tr} P_i Q_j}{\operatorname{Tr} A^s B^{1-s}}.$$

Note that the support of p_s is the same for all $s \in \mathbb{R}$. A straightforward computation yields that

$$\psi''(s) = \sum_{ij} f(i, j)^2 p_s(i, j) - \left(\sum_{ij} f(i, j) p_s(i, j) \right)^2,$$

which is 0 if and only if f is constant on the support of p_s , i.e. there exists a constant $c \in \mathbb{R}$ such that $\log \lambda_i - \log \eta_j = c$ or equivalently $\lambda_i = \delta \eta_j$ with $\delta := e^c$ for all i, j such that $\operatorname{Tr} P_i Q_j \neq 0$. Now if both $\operatorname{Tr} P_i Q_j \neq 0$ and $\operatorname{Tr} P_{i'} Q_j \neq 0$ then $\lambda_i = \delta \eta_j = \lambda_{i'}$, hence $i = i'$. The same way $\operatorname{Tr} P_i Q_j \neq 0$ and $\operatorname{Tr} P_i Q_{j'} \neq 0$ imply $j = j'$, and the rest of the statement follows.

Finally, the equivalence of (iv) and (iii) in the commutative case is easy to see. \square

Corollary 3.3. If ρ and σ are shift-invariant product states then (20) holds for all $a \in \mathbb{R}$.

Proof. First note that assumptions (a1) and (a2) are satisfied in this case. The spectral decompositions $\hat{\rho}_1 = \sum_{i \in I_1} \lambda_i P_i$ and $\hat{\sigma}_1 = \sum_{j \in J_1} \eta_j Q_j$ induce decompositions $\hat{\rho}_n = \sum_{\underline{i} \in I_1^n} \lambda_{\underline{i}} P_{\underline{i}}$ and $\hat{\sigma}_n = \sum_{\underline{j} \in J_1^n} \eta_{\underline{j}} Q_{\underline{j}}$ as in (22) for all $n \in \mathbb{N}$, where $\lambda_{\underline{i}} := \lambda_{i_1} \cdots \lambda_{i_n}$, $P_{\underline{i}} := P_{i_1} \otimes \cdots \otimes P_{i_n}$ and similarly for $\eta_{\underline{j}}$ and $Q_{\underline{j}}$. As a consequence, for the associated classical probabilities we have $p_n = p_1^{\otimes n}$, $q_n = q_1^{\otimes n}$. Then $\psi(s) = \log \sum_{(i,j) \in I_1 \times J_1} p_1(i, j)^s q_1(i, j)^{1-s}$ and

$$\liminf_n \frac{1}{n} \log e_n(a) \geq \liminf_n \frac{1}{n} \log \sum_{(\underline{i}, \underline{j}) \in I_1^n \times J_1^n} \min\{e^{-na} p_1^{\otimes n}(\underline{i}, \underline{j}), q_1^{\otimes n}(\underline{i}, \underline{j})\}. \quad (24)$$

Now we distinguish two cases. If ψ is not affine then by (ii) of Lemma 3.2 we have $\psi''(s) > 0$ for all $s \in \mathbb{R}$, and this implies that $\psi'(s)$ is in the interior of $\{x \in \mathbb{R} : \psi^*(x) < +\infty\}$ for all $s \in \mathbb{R}$. As a consequence, ψ^* is continuous at $\psi'(0)$ and at $\psi'(1)$, and this case is covered by Theorem 3.1.

Assume now that ψ is affine; then by (iv) of Lemma 3.2 we have $p_1(i, j) = \delta q_1(i, j)$ for all $(i, j) \in S := \operatorname{supp} p_1 \cap \operatorname{supp} q_1$, hence

$$\psi(s) = \log \sum_{(i,j) \in S} (\delta q_1(i, j))^s q_1(i, j)^{1-s} = s \log \delta + \log q_1(S),$$

and therefore

$$\varphi(a) = \begin{cases} -\log q_1(S), & a \leq \log \delta, \\ -\log q_1(S) + a - \log \delta, & a > \log \delta. \end{cases} \quad (25)$$

On the other hand,

$$\min\{e^{-na} p_1^{\otimes n}(\underline{i}, \underline{j}), q_1^{\otimes n}(\underline{i}, \underline{j})\} = \begin{cases} 0, & (\underline{i}, \underline{j}) \notin S^n, \\ q_1^{\otimes n}(\underline{i}, \underline{j}), & (\underline{i}, \underline{j}) \in S^n, a \leq \log \delta, \\ e^{-na} \delta^n q_1^{\otimes n}(\underline{i}, \underline{j}), & (\underline{i}, \underline{j}) \in S^n, a > \log \delta, \end{cases}$$

and thus

$$\frac{1}{n} \log \sum_{(\underline{i}, \underline{j}) \in I_1^n \times J_1^n} \min\{e^{-na} p_1^{\otimes n}(\underline{i}, \underline{j}), q_1^{\otimes n}(\underline{i}, \underline{j})\} = \begin{cases} \log q_1(S), & a \leq \log \delta, \\ \log q_1(S) - a + \log \delta, & a > \log \delta. \end{cases} \quad (26)$$

Formulas (24), (25) and (26) together give the desired statement. \square

3.2 Lower bound under factorization

Assume now that $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$. Then the limit $\psi(s) := \lim_n \psi_n(s)$ exists for all $s \in [0, 1]$ and ψ is continuous on $[0, 1]$, as was shown in Lemma 2.5. For each $m \in \mathbb{N}$ let

$$\varphi_m(a) := \max_{0 \leq s \leq 1} \{as - \psi(s)\}, \quad a \in \mathbb{R}$$

be the polar function of ψ_m . The bound (19) implies that

$$\varphi_m(a) - \frac{1}{m} \log \eta \leq \varphi(a) \leq \varphi_m(a) + \frac{1}{m} \log \eta \quad (27)$$

for every $a \in \mathbb{R}$ and $m \in \mathbb{N}$, hence φ is the uniform limit of the sequence $\{\varphi_m\}$.

Theorem 3.4. Let $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$. Then for any $a \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \min_{0 \leq T_n \leq I_n} \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} = -\varphi(a). \quad (28)$$

Proof. Due to Lemma 2.2 it suffices to prove that (21) holds for any sequence of tests $\{T_n\}$. We can assume that ρ and σ have the same factorization constant η . Let $m \in \mathbb{N}$ be fixed and write $n > m$ in the form $n = (k-1)m + r$ with $1 \leq r \leq m$. With $\gamma := \min\{1, e^{(m-r)a}; 0 \leq r < m\}$ we have

$$\begin{aligned} e^{-na} \alpha_n(T_n) + \beta_n(T_n) &\geq \gamma (e^{-kma} \alpha_n(T_n) + \beta_n(T_n)) \\ &= \gamma (e^{-kma} \alpha_{km}(T_n \otimes I_{[n+1, km]}) + \beta_{km}(T_n \otimes I_{[n+1, km]})) \\ &\geq \gamma \eta^{-(k-1)} \min_{0 \leq T \leq I} \{e^{-kma} \text{Tr}[\hat{\rho}_m^{\otimes k}(I - T)] + \text{Tr}[\hat{\sigma}_m^{\otimes k} T]\}, \end{aligned} \quad (29)$$

where we used lower factorization property in the last step. Let

$$\Psi_m(s) := \log \text{Tr} \hat{\rho}_m^s \hat{\sigma}_m^{1-s} = m\psi_m(s) \quad \text{and} \quad \Phi_m(a) := \max_{0 \leq s \leq 1} \{as - \Psi_m(s)\} = m\varphi_m(a/m).$$

By Corollary 3.3 we have

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \log \min_{0 \leq T \leq I} \{e^{-kma} \text{Tr}[\hat{\rho}_m^{\otimes k}(I - T)] + \text{Tr}[\hat{\sigma}_m^{\otimes k} T]\} \geq -\Phi_m(ma).$$

Combining it with (29) we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} &\geq -\frac{1}{m} \log \eta - \frac{1}{m} \Phi_m(ma) = -\frac{1}{m} \log \eta - \varphi_m(a) \\ &\geq -\frac{1}{m} \log \eta - \varphi(a) - \frac{1}{m} \log \eta, \end{aligned}$$

where we used (27) in the last inequality. Taking the limit in m gives the assertion. \square

3.3 Some remarks

Note that equations (20) and (28) can be reformulated as

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_{\min}(\rho_n : \sigma_n | \pi_n) = -\max\{\varphi(a), \varphi(a) - a\} = \begin{cases} -\varphi(a) & \text{if } a \geq 0, \\ -\varphi(a) + a & \text{if } a < 0, \end{cases} \quad (30)$$

with $\pi_n := \frac{e^{-na}}{1+e^{-na}}$, therefore giving an extension of the theorem for the Chernoff bound to a setting when the prior probabilities are not constant, but depend on n in the given way. From Lemma 2.2 we get that (30) holds whenever $a \in A(\vec{\rho}, \vec{\sigma})$, where $A(\vec{\rho}, \vec{\sigma})$ denotes the set of all $a \in \mathbb{R}$ for which inequality (21) is satisfied. In particular, if $0 \in A(\vec{\rho}, \vec{\sigma})$ then we recover the theorem for the Chernoff bound [2, 10]. This is the case e.g. when $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$ (since $A(\vec{\rho}, \vec{\sigma}) = \mathbb{R}$ by Theorem 3.4) and when assumptions (a1) and (a2) are satisfied and $0 \neq \partial^+ \psi(0)$ and $0 \neq \partial^- \psi(1)$ (since $A(\vec{\rho}, \vec{\sigma}) \supset \mathbb{R} \setminus \{\partial^+ \psi(0), \partial^- \psi(1)\}$ by Theorem 3.1).

Alternatively, one can interpret Theorems 3.1 and 3.4 as the theorem for the Chernoff bound in the setting when hypothesis testing is performed between the states σ_n and the unnormalized states $e^{-na} \rho_n$. Indeed, in the setting of Section 3.1 or Section 3.2 we have

$$\psi_a(s) := \lim_n \frac{1}{n} \log \text{Tr} \left(e^{-na} \hat{\rho}_n \right)^s \hat{\sigma}_n^{1-s} = -\{as - \psi(s)\}, \quad s \in [0, 1],$$

and

$$\lim_n \frac{1}{n} \log P_{\min}(e^{-na} \rho_n : \sigma_n | \pi) = \min_{0 \leq s \leq 1} \psi_a(s)$$

for any $\pi \in (0, 1)$ and $a \in A(\vec{\rho}, \vec{\sigma})$.

4 The Hoeffding bound and related exponents

Our main goal in this section is to derive the theorem for the Hoeffding bound in the settings of Sections 3.1 and 3.2. To treat the two settings in a unified way, we derive all our results under the following assumptions:

(A1) The limit $\psi(s) := \lim_{n \rightarrow \infty} \psi_n(s)$ exists as a real number for all $s \in [0, 1]$ and ψ is continuous on $[0, 1]$.

(A2) The inequality

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \min_{0 \leq T_n \leq I_n} \{e^{-na} \alpha_n(T_n) + \beta_n(T_n)\} \geq -\varphi(a) \quad (31)$$

holds for all $a < \partial^- \psi(1)$ (the left derivative of ψ at 1), except possibly for finitely many values of a , where φ is given in (10).

Though assumptions (A1) and (A2) are admittedly rather artificial, they have the merits that they are satisfied in the cases of our interest on the one hand (see Section 3),

and on the other hand they give the minimal requirements under which the results of this section are valid, thus providing a better view on the logical relations among our results.

We begin by introducing $\tilde{\psi}(s) := \psi(1 - s)$, $s \in [0, 1]$ and its polar function

$$\tilde{\varphi}(a) := \max_{0 \leq s \leq 1} \{as - \tilde{\psi}(s)\} = a + \varphi(-a), \quad a \in \mathbb{R}.$$

We also define

$$\hat{\varphi}(a) := \tilde{\varphi}(-a) = \varphi(a) - a, \quad a \in \mathbb{R}. \quad (32)$$

In Appendix C we give an illustration of the above definitions and the properties listed in the following:

Lemma 4.1. The functions φ and $\hat{\varphi}$ have the following properties:

- (i) φ is convex, continuous, and increasing on \mathbb{R} . Moreover, it is constant $-\psi(0)$ on the interval $(-\infty, \partial^+ \psi(0)]$ and strictly increasing on $(\partial^+ \psi(0), +\infty)$.
- (ii) $\hat{\varphi}$ is convex, continuous, and decreasing on \mathbb{R} . Moreover, it is strictly decreasing on the interval $(-\infty, \partial^- \psi(1))$ and is constant $-\psi(1)$ on the interval $[\partial^- \psi(1), +\infty)$.

Proof. All properties follow immediately from the very definitions of φ and $\hat{\varphi}$, except for strict monotonicity. We only prove it for $\hat{\varphi}$, as the proof for φ is completely similar. Note that

$$\hat{\varphi}(a) = \varphi(a) - a = \max_{0 \leq s \leq 1} \{a(s - 1) - \psi(s)\},$$

and let $s_a := \arg \max_{0 \leq s \leq 1} \{a(s - 1) - \psi(s)\}$. It follows from $a(1 - 1) - \psi(1) = -\psi(1)$ that if a is such that $\varphi(a) - a > -\psi(1)$ (i.e. $a < \partial^- \psi(1)$) then $s_a < 1$. Hence for $b < a$

$$\varphi(a) - a = a(s_a - 1) - \psi(s_a) < b(s_a - 1) - \psi(s_a) \leq \varphi(b) - b.$$

□

Remark 4.2. Due to the above listed properties, one can extend φ and $\hat{\varphi}$ to continuous and monotonic functions on $\mathbb{R} \cup \{+\infty\}$ by defining $\varphi(+\infty) := +\infty$ and $\hat{\varphi}(+\infty) := -\psi(1)$.

Remark 4.3. Note that $\psi_n(s) \leq 0$ for all $s \in [0, 1]$, and $n \in \mathbb{N}$, hence the same holds for ψ . If $\text{supp } \hat{\rho}_n \leq \text{supp } \hat{\sigma}_n$ then $\psi_n(1) = \frac{1}{n} \log \text{Tr} [\hat{\rho}_n \text{supp } \hat{\sigma}_n] = 0$, and a straightforward computation shows that

$$\partial^- \psi_n(1) = \frac{1}{n} S(\rho_n \| \sigma_n),$$

where $S(\rho_n \| \sigma_n) := \text{Tr } \hat{\rho}_n (\log \hat{\rho}_n - \log \hat{\sigma}_n)$ is the *relative entropy* of the states ρ_n and σ_n . Convexity of ψ_n implies

$$\psi_n(s) \geq \psi_n(1) + (s - 1) \partial^- \psi_n(1) = (s - 1) \frac{1}{n} S(\rho_n \| \sigma_n), \quad s \in [0, 1]. \quad (33)$$

Assume now that $\text{supp } \hat{\rho}_n \leq \text{supp } \hat{\sigma}_n$ holds for all large enough n , and that the mean relative entropy

$$S_M(\vec{\rho} \| \vec{\sigma}) := \lim_{n \rightarrow \infty} \frac{1}{n} S(\rho_n \| \sigma_n)$$

exists. (Note that if $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$ then if $\text{supp } \hat{\rho}_m \subset \text{supp } \hat{\sigma}_m$ for some $m \in \mathbb{N}$ then it also holds for all $n \in \mathbb{N}$, and the mean relative entropy exists even if one requires only the upper factorization property to hold [13, Theorem 2.1].) Then $\psi(1) = 0$, and taking the limit in (33) yields

$$\psi(s) \geq (s-1) S_{\text{M}}(\rho \parallel \sigma), \quad s \in [0, 1],$$

therefore

$$\partial^- \psi(1) \leq S_{\text{M}}(\rho \parallel \sigma). \quad (34)$$

Similarly, if we replace the condition $\text{supp } \hat{\rho}_n \leq \text{supp } \hat{\sigma}_n$ with $\text{supp } \hat{\rho}_n \geq \text{supp } \hat{\sigma}_n$ in the above argument then we get $\psi(0) = 0$ and $\partial^+ \psi(0) \geq -S_{\text{M}}(\sigma \parallel \rho)$.

Note that if $\rho_n = \rho_1^{\otimes n}$ and $\sigma_n = \sigma_1^{\otimes n}$, $n \in \mathbb{N}$, with $\text{supp } \hat{\rho}_1 \wedge \text{supp } \hat{\sigma}_1 \neq 0$ and $\text{supp } \hat{\rho}_1 \not\subseteq \text{supp } \hat{\sigma}_1$, then $\partial^- \psi(1)$ is finite while $S_{\text{M}}(\rho \parallel \sigma) = +\infty$, hence (34) cannot be expected to hold as an equality in general. In Appendix B we show examples for correlated states ρ, σ on a spin chain for which $\partial^- \psi(1) = S_{\text{M}}(\vec{\rho} \parallel \vec{\sigma})$ can be shown by an explicit computation.

Lemma 4.4. Let $a < \partial^- \psi(1)$. Then for any sequence of tests $\{T_n\}$ satisfying

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(T_n) \leq -\{\varphi(a) - a\},$$

we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \geq -\varphi(a).$$

Proof. We follow the same argument as in [17]. For any b satisfying (31) we have

$$\begin{aligned} -\varphi(b) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \{e^{-nb} \alpha_n(T_n) + \beta_n(T_n)\} \\ &\leq \max \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n), -b + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(T_n) \right\} \\ &\leq \max \left\{ \liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n), -b - \varphi(a) + a \right\} \end{aligned}$$

If $a < b < \partial^- \psi(1)$, then $-\varphi(a) + a < -\varphi(b) + b$ by Lemma 4.1 (ii), hence

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \geq -\varphi(b).$$

Continuity of φ then yields

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \geq -\varphi(a).$$

□

Lemmas 4.4 and 2.2 give the following:

Corollary 4.5. For $a < \partial^- \psi(1)$ and $S_{n,a} := \{e^{-na} \hat{\rho}_n - \hat{\sigma}_n > 0\}$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(S_{n,a}) = -\varphi(a).$$

Remark 4.6. Recall that φ is strictly increasing on the interval $[\partial^+ \psi(0), +\infty)$. An obvious modification of the above proof then yields that for any $a > \partial^+ \psi(0)$ and any sequence of tests $\{T_n\}$ satisfying

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \leq -\varphi(a)$$

we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(T_n) \geq -\{\varphi(a) - a\}.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(S_{n,a}) = -\{\varphi(a) - a\}$$

for any $a > \partial^+ \psi(0)$.

Note that $\partial^+ \psi(0) \leq \partial^- \psi(1)$ due to the convexity of ψ , and the interval

$$I_\psi := \{a \in \mathbb{R} : \partial^+ \psi(0) < a < \partial^- \psi(1)\}$$

is nonempty if and only if the graph of ψ is not a straight line segment. Corollary 4.5 and Remark 4.6 give the following:

Theorem 4.7. For $a \in I_\psi$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_n(S_{n,a}) &= -\{\varphi(a) - a\}, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(S_{n,a}) &= -\varphi(a). \end{aligned}$$

Now we are in a position to prove our main result.

Theorem 4.8. For any $r \in \mathbb{R}$ we have

$$\underline{B}(r|\vec{\rho}||\vec{\sigma}) = \overline{B}(r|\vec{\rho}||\vec{\sigma}) = B(r|\vec{\rho}||\vec{\sigma}) = - \sup_{0 \leq s < 1} \frac{-sr - \psi(s)}{1 - s}.$$

For $r < -\psi(1)$ all the above quantities are equal to $-\infty$; for $r \geq -\psi(1)$ we have

$$B(r|\vec{\rho}||\vec{\sigma}) = - \max_{0 \leq s < 1} \frac{-sr - \psi(s)}{1 - s} = -\varphi(a_r),$$

where a_r is a unique number in $(-\infty, \partial^- \psi(1)]$ such that $\hat{\varphi}(a_r) = r$.

Proof. First assume that $r < -\psi(1)$. A straightforward computation shows that $\frac{-sr-\psi(s)}{1-s}$ tends to $+\infty$ as $s \nearrow 1$, hence $\sup_{0 \leq s < 1} \frac{-sr-\psi(s)}{1-s} = +\infty$. Let $T_n := I - \text{supp } \hat{\sigma}_n$. Then $\frac{1}{n} \log \alpha_n(T_n) = \psi_n(1)$ by definition, hence $\limsup_n \frac{1}{n} \log \alpha_n(T_n) = \psi(1) < -r$ and obviously $\beta_n(T_n) = 0$ for all n . Hence (by using the convention $\log 0 := -\infty$) we get

$$B(r|\vec{\rho}||\vec{\sigma}) \leq -\infty = -\sup_{0 \leq s < 1} \frac{-sr - \psi(s)}{1-s},$$

and the inequalities in (13) give the desired statement.

Now if $r \geq -\psi(1)$ then the properties of $\hat{\varphi}$ listed in (ii) of Lemma 4.1 guarantee the existence of a unique $a_r \leq \partial^- \psi(1)$ such that $\hat{\varphi}(a_r) = r$ (see Figure 2 in Appendix C for an illustration). Note that if $\partial^+ \psi(1) = +\infty$ then $a_r = +\infty$ and we use the conventions of Remark 4.2. If $\limsup_n \frac{1}{n} \log \alpha_n(T_n) < -r$ then there exists a $b < a_r$ such that for all $b < a < a_r$ we have

$$\limsup_n \frac{1}{n} \log \alpha_n(T_n) \leq -\{\varphi(a) - a\} < -\{\varphi(a_r) - a_r\} = -r.$$

By Lemma 4.4

$$\liminf_n \frac{1}{n} \log \beta_n(T_n) \geq -\varphi(a),$$

and by taking the limit $a \nearrow a_r$ we obtain

$$\liminf_n \frac{1}{n} \log \beta_n(T_n) \geq -\varphi(a_r).$$

Hence

$$\underline{B}(r|\vec{\rho}||\vec{\sigma}) \geq -\varphi(a_r).$$

On the other hand, by Lemma 2.2 and Corollary 4.5 we have for any $a < a_r$

$$\begin{aligned} \limsup_n \frac{1}{n} \log \alpha_n(S_{n,a}) &\leq -\{\varphi(a) - a\} < -\{\varphi(a_r) - a_r\} = -r, \\ \lim_n \frac{1}{n} \log \beta_n(S_{n,a}) &= -\varphi(a), \end{aligned}$$

hence $B(r|\vec{\rho}||\vec{\sigma}) \leq -\varphi(a)$. Now taking $a \nearrow a_r$ we get

$$B(r|\vec{\rho}||\vec{\sigma}) \leq -\varphi(a_r).$$

Taking the inequalities in (13) into account, we have

$$-\varphi(a_r) \leq \underline{B}(r|\vec{\rho}||\vec{\sigma}) \leq \overline{B}(r|\vec{\rho}||\vec{\sigma}) \leq B(r|\vec{\rho}||\vec{\sigma}) \leq -\varphi(a_r).$$

To prove the last identity assume first that $r > -\psi(1)$. Then $a_r < \partial^- \psi(1)$ and $s_r < 1$, where $s_r := \arg \max_{0 \leq s \leq 1} \{a_r s - \psi(s)\}$. Thus we have

$$r = \varphi(a_r) - a_r = a_r s_r - \psi(s_r) - a_r \geq a_r s - \psi(s) - a_r, \quad 0 \leq s \leq 1,$$

hence $a_r \geq \frac{-r-\psi(s)}{1-s}$ for any $0 \leq s < 1$ with equality for $s = s_r$. Then

$$\varphi(a_r) = a_r s_r - \psi(s_r) \geq a_r s - \psi(s) \geq \frac{-sr - \psi(s)}{1-s}, \quad 0 \leq s < 1, \quad (35)$$

and equality holds for $s = s_r$.

Now if $r = -\psi(1)$ then $a_r = \partial^- \psi(1)$ and $\varphi(a_r) = \partial^- \psi(1) - \psi(1)$, and one can easily see that

$$\lim_{s \nearrow 1} \frac{-s(-\psi(1)) - \psi(s)}{1-s} = \partial^- \psi(1) - \psi(1).$$

If $\partial^- \psi(1) = +\infty$ then this gives the desired identity immediately. If $\partial^- \psi(1) < +\infty$ then the statement follows from the fact that inequalities in (35) are still valid (with $s_r = 1$). \square

One can get a certain interpolation between the setting of Stein's lemma and the theorem for the Hoeffding bound by requiring that the α_n 's converge to zero exponentially, but without constraint on the value of the exponent. The corresponding exponents for the β_n 's are $\underline{B}(0|\vec{\rho}||\vec{\sigma})$, $\overline{B}(0|\vec{\rho}||\vec{\sigma})$ and $B(0|\vec{\rho}||\vec{\sigma})$. We have the following:

Proposition 4.9. If $\psi(1) = 0$ then

$$\underline{B}(0|\vec{\rho}||\vec{\sigma}) = \overline{B}(0|\vec{\rho}||\vec{\sigma}) = B(0|\vec{\rho}||\vec{\sigma}) = -\partial^- \psi(1).$$

Moreover, if ρ and σ satisfy the upper factorization property and $\text{supp } \hat{\rho}_n \leq \text{supp } \hat{\sigma}_n$ for all $n \in \mathbb{N}$ then

$$B(0|\vec{\rho}||\vec{\sigma}) \geq -S_M(\rho||\sigma).$$

Proof. The first statement is a special case of Theorem 4.8. To see this, take $r = -\psi(1) = 0$; then $a_r = \partial^- \psi(1)$ and $\varphi(a_r) = \partial^- \psi(1)$. The second statement follows from (34). \square

In studying Stein's lemma, one is interested in the exponents $B(\vec{\rho}||\vec{\sigma})$, $\underline{B}(\vec{\rho}||\vec{\sigma})$ and $\overline{B}(\vec{\rho}||\vec{\sigma})$, where

$$B(\vec{\rho}||\vec{\sigma}) := \inf_{\{T_n\}} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n) \mid \lim_{n \rightarrow \infty} \alpha_n(T_n) = 0 \right\},$$

and $\underline{B}(\vec{\rho}||\vec{\sigma})$ and $\overline{B}(\vec{\rho}||\vec{\sigma})$ are defined similarly, by taking $\liminf_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n)$ and $\limsup_{n \rightarrow \infty} \frac{1}{n} \log \beta_n(T_n)$. Obviously, $\underline{B}(\vec{\rho}||\vec{\sigma}) \leq \overline{B}(\vec{\rho}||\vec{\sigma}) \leq B(\vec{\rho}||\vec{\sigma})$, and Theorem 4.8 has the following consequence:

Proposition 4.10. If $\psi(1) < 0$ then

$$\underline{B}(\vec{\rho}||\vec{\sigma}) = \overline{B}(\vec{\rho}||\vec{\sigma}) = B(\vec{\rho}||\vec{\sigma}) = -\infty = -S_M(\rho||\sigma).$$

If $\psi(1) = 0$ then

$$B(\vec{\rho}||\vec{\sigma}) \leq -\partial^- \psi(1).$$

Proof. If $\psi(1) < 0$ then there exists an $N \in \mathbb{N}$ such that $\text{Tr } \hat{\rho}_n[\text{supp } \hat{\sigma}_n] < 1$ for all $n \geq N$, i.e. $\text{supp } \hat{\rho}_n \not\subseteq \text{supp } \hat{\sigma}_n$ and hence $S(\rho_n || \sigma_n) = +\infty$ for all $n \geq N$, implying $S_M(\rho || \sigma) = +\infty$. The rest of the statements follow immediately from the fact that $B(\vec{\rho} || \vec{\sigma}) \leq B(0 || \vec{\rho} || \vec{\sigma})$, and that $B(0 || \vec{\rho} || \vec{\sigma}) = -\infty$ when $\psi(1) < 0$ by Theorem 4.8 and $B(0 || \vec{\rho} || \vec{\sigma}) = -\partial^- \psi(1)$ when $\psi(1) = 0$ by Proposition 4.9. \square

Note that when $\rho = \rho_1^{\otimes \infty}$ and $\sigma = \sigma_1^{\otimes \infty}$ are product states with $\text{supp } \hat{\rho}_1 \leq \text{supp } \hat{\sigma}_1$ then $\psi(1) = 0$ and $\partial^- \psi(1) = S(\rho_1 || \sigma_1)$, and we get back the well-known formula for the direct part of Stein's lemma.

5 Concluding remarks

We have studied various error exponents, including the Chernoff and Hoeffding bounds in a binary (asymptotic) hypothesis testing problem. While following a rather general formulation, the main applicability of our results is the hypothesis testing problem for two states on a spin chain, both satisfying the factorization properties given in Definition 2.4. That the study of such states is sufficiently well motivated was established in [10], where we have shown that the factorization properties are satisfied by the global Gibbs states of translation-invariant finite-range interactions. Other important examples for states to which our results may be applicable are provided by the Markov-type class of finitely correlated states [6] (see e.g. Example B.1 and [10]). While finitely correlated states always satisfy the upper factorization property [10], it is an open question at the moment to find necessary and sufficient conditions for the lower factorization property to hold. As Remark B.2 suggests, it may be possible to prove the validity of assumptions (a1) and (a2) for finitely correlated states even if the lower factorization property fails to hold. Similar conditions to our factorization properties were used at various places in the literature. Probably the closest to our factorization properties is the *-mixing condition (see e.g. [4] and references therein). However, the relation among these conditions and the factorization properties in the quantum setting is an open question at the moment.

Our main tool in deriving the upper bounds in Lemma 2.2 was the powerful trace inequality (7) discovered in [1], that was successfully applied to give the Chernoff [1] and Hoeffding [8] upper bounds in the i.i.d. case. Our Corollary 2.3 is an extension of [8, Theorem 1] to the very general setting of Section 2.1, and a slight simplification as well, as the tests only depend on the parameter a , and not on s as in [8].

The main idea in deriving the lower bound (21) in Section 3.1 is to relate the quantum problem to a classical hypothesis testing problem by the method of [20] and then use large deviation techniques to treat the classical problem. This approach was used in the i.i.d. case to derive the quantum Chernoff [20] and Hoeffding [17] lower bounds. In [17] the two states were implicitly assumed to have the same support, which assumption can easily be removed by restricting the classical probability distributions onto the intersection of their supports; this approach was followed in [2]. In Section 3.1 we have followed a different way to circumvent the restriction of equivalent supports, by slightly modifying the way to assign classical measures to the original states. In the non-i.i.d. case it is a natural choice to use the Gärtner-Ellis theorem to establish the lower bound in the classical hypothesis testing problem, and the differentiability condition in assumption (a2) is essentially the

requirement of the differentiability of the logarithmic moment generating function in the Gärtner-Ellis theorem. As we argue in Section 3.3, the main results of Section 3, Theorems 3.1 and 3.4 are essentially giving the theorem for the Chernoff bound in an appropriate setting. The fact that Corollary 3.3 is true for all real numbers a seems to be new even in the i.i.d. setting. The exclusion of the cases $a = \partial^+ \psi(0)$ and $a = \partial^- \psi(1)$ in Theorem 3.1 is strongly related to the possibility of a pathological situation when the graph of ψ becomes a straight line, and could possibly be removed if a similar characterization to that in Lemma 3.2 was available also in the non-i.i.d. case.

It is well-known in the information spectrum approach that the limits of the quantities $\frac{1}{n} \log \alpha_n(S_{n,a})$ and $\frac{1}{n} \log \beta_n(S_{n,a})$ are strongly related to the theorem for the Hoeffding bound, as was emphasized e.g. in [7] and [19]. In Lemma 4.4 we follow the way of [17] to derive these limits from the Chernoff-type theorems of Section 3. Theorem 4.7 was stated as a conjecture in [17] and it was proven shortly after in [18] in the i.i.d. setting for the values of a between $-S_M(\sigma_1 \parallel \rho_1)$ and $S_M(\rho_1 \parallel \sigma_1)$; this coincides with our I_ψ when $\text{supp } \hat{\rho}_1 = \text{supp } \hat{\sigma}_1$ is assumed in the i.i.d. setting. The importance of the above limits are clearly shown by the fact that the results of Theorem 4.8 and Propositions 4.9 and 4.10 hold true whenever Corollary 4.5 is true (here we benefit from the fact that Lemma 2.2 is unconditionally true in the most general setting, showing again the power of inequality (7)).

The interpretation of $Q(\rho, \sigma) := \min_{0 \leq s \leq 1} \text{Tr } \hat{\rho}^s \hat{\sigma}^{1-s}$ as a distinguishability measure on the state space of a finite dimensional quantum system was investigated in [1], where a detailed analysis of its properties and its relation to other measures (like fidelity, trace distance and relative entropy) was given. Here we would like to stress the importance of its negative logarithmic version

$$C(\rho, \sigma) := - \min_{0 \leq s \leq 1} \log \text{Tr } \hat{\rho}^s \hat{\sigma}^{1-s} = - \log Q(\rho, \sigma).$$

It is jointly convex in its variables (due to Lieb's concavity theorem), monotonic decreasing under 2-positive trace-preserving maps [23, 24], and easily seen to be strictly positive, thus sharing some of the most important properties of relative entropy. Moreover, if $\vec{\rho}, \vec{\sigma} \in \mathcal{S}_{\text{fact}}(\mathcal{C})$ then the uniform convergence established in Lemma 2.5 shows that the limit

$$C_M(\rho, \sigma) := \lim_{n \rightarrow \infty} \frac{1}{n} C(\rho_n, \sigma_n) \tag{36}$$

exists (and coincides with $\varphi(0)$), further extending the analogy with the relative entropy. Theorem 3.4 for the Chernoff bound gives that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \min_{0 \leq T_n \leq I} \{\alpha_n(T_n) + \beta_n(T_n)\} = -C_M(\rho, \sigma),$$

thus giving an operational interpretation to the mean Chernoff distance, and showing that it plays exactly the same role in the symmetric setting of the theorem for the Chernoff bound as the mean relative entropy plays in the asymmetric setting of Stein's lemma. Obviously, the asymptotic quantity (36) is still jointly concave and monotonic decreasing under 2-positive trace-preserving maps; it is not clear, however, whether the strict positivity property is preserved under taking the limit.

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Appendix A

Note that Lemma 2.1 becomes trivial when $\mathcal{B}(\mathcal{H})$ is replaced with a commutative C^* -algebra. Indeed, in this case elements of the algebra are functions on some compact space X , and if f, g are non-negative functions then

$$\frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|) = \min\{f(x), g(x)\} \leq f(x)^s g(x)^{1-s}, \quad s \in [0, 1],$$

and the trace can be replaced with integration with respect to an arbitrary positive measure. The minimum of two nonnegative operators $A, B \in \mathcal{B}(\mathcal{H})$ might be defined as the unique self-adjoint $X \in \mathcal{B}(\mathcal{H})$ for which

$$X \leq A, \quad X \leq B \quad \text{and} \quad Y \leq A, \quad Y \leq B \Rightarrow Y \leq X$$

holds. Note, however, that such an operator does not exist in general, even when A and B commute with each other. On the other hand, the following is true:

Proposition A.1. For any nonnegative operators $A, B \in \mathcal{B}(\mathcal{H})$, we have

$$\max\{\text{Tr } X : X \leq A, X \leq B\} \leq \text{Tr } A^s B^{1-s}, \quad s \in [0, 1].$$

The proof follows immediately from Lemma 2.1 and the following lemma, which is a special case of the duality theorem in multiple hypothesis testing [25, 14]. Since the proof of the binary case is immediate, we include it for readers' convenience.

Lemma A.2. For any nonnegative operators $A, B \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} \max\{\text{Tr } X : X \leq A, X \leq B\} &= \min_{0 \leq T \leq I} \{\text{Tr } A(I - T) + \text{Tr } BT\} \\ &= \text{Tr } \frac{1}{2}(A + B - |A - B|). \end{aligned}$$

Proof. Suppose that $X \leq A, X \leq B$. Then $X = X^*$ and for any operator T satisfying $0 \leq T \leq I$ we have

$$\text{Tr } X = \text{Tr } X(I - T) + \text{Tr } XT \leq \text{Tr } A(I - T) + \text{Tr } BT.$$

Conversely, let $S := \{A - B > 0\}$ and $X := A(I - S) + BS = \frac{1}{2}(A + B - |A - B|)$. Then

$$X = A - (A - B)_+ \leq A, \quad X = B - (A - B)_- \leq B,$$

where $(A - B)_+$ and $(A - B)_-$ denote the positive and the negative parts of $A - B$, respectively. \square

Loosely speaking, the above shows that as long as Tr is taken, $\frac{1}{2}(A + B - |A - B|)$ can be considered as the minimum of A and B . Note, however, that $\frac{1}{2}(A + B - |A - B|)$ need not even be positive semidefinite when both A and B are positive semidefinite. A simple counterexample is given by $A := |a\rangle\langle a|$ and $B := |b\rangle\langle b|$ with $a = (1, i)$ and $b = (1, 1)$.

Appendix B

Unfortunately, assumptions (a1) and (a2) in Section 3.1 seem to be difficult to verify in general correlated cases. Below we show a specific example for a pair of correlated states on a spin chain for which assumptions (a1) and (a2) can directly be verified. Note that the example is non-classical in the sense that the local densities $\hat{\rho}_n$ and $\hat{\sigma}_n$ need not commute with each other. However, both ρ and σ exhibit only classical correlations among the sites of the chain, i.e. all local densities $\hat{\rho}_n$, $\hat{\sigma}_n$, $n \in \mathbb{N}$, are separable.

Example B.1. Let $T = \{T_{xy}\}_{x,y \in \mathcal{X}}$ and $S = \{S_{xy}\}_{x,y \in \mathcal{X}}$ be irreducible stochastic matrices with corresponding faithful stationary distributions $r = \{r_x\}_{x \in \mathcal{X}}$ and $p = \{p_x\}_{x \in \mathcal{X}}$ on some finite set \mathcal{X} . Moreover, let $\{\vartheta_{xy}\}_{x,y \in \mathcal{X}}$ and $\{\varphi_{xy}\}_{x,y \in \mathcal{X}}$ be sets of states on a finite-dimensional C^* -algebra \mathcal{A} and $\Theta_x := \sum_y T_{xy} \vartheta_{xy}$, $\Phi_x := \sum_y S_{xy} \varphi_{xy}$. The local states

$$\begin{aligned} \rho_n &:= \sum_{x_1, \dots, x_n \in \mathcal{X}} r_{x_1} (T_{x_1 x_2} \vartheta_{x_1 x_2}) \otimes \dots \otimes (T_{x_{n-1} x_n} \vartheta_{x_{n-1} x_n}) \otimes \Theta_{x_n}, \\ \sigma_n &:= \sum_{x_1, \dots, x_n \in \mathcal{X}} p_{x_1} (S_{x_1 x_2} \varphi_{x_1 x_2}) \otimes \dots \otimes (S_{x_{n-1} x_n} \varphi_{x_{n-1} x_n}) \otimes \Phi_{x_n}, \end{aligned}$$

are easily seen to extend to translation-invariant states ρ and σ on the spin chain $\mathcal{C} = \otimes_{k \in \mathbb{Z}} \mathcal{A}$. (Actually, ρ and σ are ergodic finitely correlated states with a commutative auxiliary algebra in their generating triples; see [6] and also [10] for details.)

Let us assume that there exists a set of non-zero projections $\{P_x\}_{x \in \mathcal{X}}$ in \mathcal{A} with orthogonal ranges such that

$$\text{supp } \hat{\vartheta}_{xy} \vee \text{supp } \hat{\varphi}_{xy} \leq P_x \quad \text{and also} \quad \text{supp } \hat{\vartheta}_{xy} \wedge \text{supp } \hat{\varphi}_{xy} \neq 0, \quad x, y \in \mathcal{X}. \quad (37)$$

Then

$$\begin{aligned} \text{Tr } \hat{\rho}_n^s \hat{\sigma}_n^{1-s} &= \sum_{x_1, \dots, x_n} r_{x_1}^s p_{x_1}^{1-s} \left(\text{Tr } \hat{\Theta}_{x_n}^s \hat{\Phi}_{x_n}^{1-s} \right) \prod_{k=1}^{n-1} \left(T_{x_k x_{k+1}}^s S_{x_k x_{k+1}}^{1-s} \text{Tr } \hat{\vartheta}_{x_k x_{k+1}}^s \hat{\varphi}_{x_k x_{k+1}}^{1-s} \right) \\ &= \langle a(s), Q(s)^{n-1} b(s) \rangle \end{aligned}$$

for every $s \in \mathbb{R}$ with

$$a(s)_x := r_x^s p_x^{1-s}, \quad b(s)_x := \text{Tr } \hat{\Theta}_x^s \hat{\Phi}_x^{1-s} \quad \text{and} \quad Q(s)_{x,y} := T_{xy}^s S_{xy}^{1-s} \text{Tr } \hat{\vartheta}_{xy}^s \hat{\varphi}_{xy}^{1-s}.$$

Now if $Q(s)$ is irreducible for some (and hence for all) $s \in \mathbb{R}$ then by the Perron-Frobenius theorem we have

$$\psi(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \langle a(s), Q(s)^{n-1} b(s) \rangle = \log r(s), \quad s \in \mathbb{R},$$

where $r(s)$ is the spectral radius of $Q(s)$ (see e.g. [5, Theorem 3.1.1]). Being a simple eigenvalue, the function $s \mapsto r(s)$ is smooth (cf. [15]), and so is ψ , hence assumptions (a1) and (a2) are satisfied in this case.

Assume now that $\text{supp } \hat{\rho}_n \leq \text{supp } \hat{\sigma}_n$ for some $n \geq 2$. This is easily seen to be equivalent to the conditions

$$T_{xy} = 0 \quad \text{if} \quad S_{xy} = 0, \quad (38)$$

$$\text{supp } \hat{\vartheta}_{xy} \leq \text{supp } \hat{\varphi}_{xy} \quad \text{if} \quad T_{xy} > 0, \quad (39)$$

and hence is independent of the value of n . (Note that the first condition states that the classical Markov chain generated by T and r is absolutely continuous with respect to that generated by S and p .) It is easily seen that in this case $Q(s)$ is irreducible for every $s \in \mathbb{R}$, hence we can apply the above argument and obtain $\psi(s) = \log r(s)$, $s \in \mathbb{R}$. Simplicity of $r(s)$ as an eigenvalue of $Q(s)$ yields that one can choose the corresponding Perron-Frobenius (left) eigenvectors $\xi(s) = \{\xi_x(s)\}_{x \in \mathcal{X}}$ to form a strictly positive probability distribution for all $s \in \mathbb{R}$ such that the function $s \mapsto \xi(s)$ is smooth (cf. [15]). Let $e := (1, \dots, 1)$ be the identity vector. Using the facts that $r(s) = \langle \xi(s)Q(s), e \rangle$, that $\langle \xi'(s), e \rangle = 0$ due to the fact that $\langle \xi(s), e \rangle = 1$ for all s , and that $Q(1) = T$, $\xi(1) = r$, $r(T) = 1$, we obtain

$$\begin{aligned} \psi'(1) &= \frac{r'(1)}{r(1)} = \langle \xi'(1), Q(1)e \rangle + \langle \xi(1), Q'(1)e \rangle \\ &= \sum_x r_x S(T_x \parallel S_x) + \sum_{xy} r_x T_{xy} S(\vartheta_{xy} \parallel \varphi_{xy}). \end{aligned}$$

A straightforward computation then shows that the latter expression is exactly the mean relative entropy $S_M(\rho \parallel \sigma)$. Similarly, if we impose the condition $\text{supp } \hat{\rho}_n \geq \text{supp } \hat{\sigma}_n$ for some (and hence for all) $n \geq 2$ then we obtain $\psi'(0) = -S_M(\sigma \parallel \rho)$.

Remark B.2. In the above construction, let \mathcal{A} be isomorphic to the function algebra on some finite set \mathcal{X} , and let $\hat{\vartheta}_{xy} := \hat{\varphi}_{xy} := 1_{\{x\}}$ (the indicator function of $\{x\}$) for all $x, y \in \mathcal{X}$. Then $\hat{\rho}_n$ and $\hat{\sigma}_n$ are the densities of the n -site restrictions of the Markov measures μ and ν generated by (T, r) and (S, p) , respectively, and the conditions (37) and (39) are automatically satisfied.

Now it is easy to see that μ and ν satisfy the lower factorization property if and only if T and R are entrywise strictly positive matrices, which condition is sufficient but not necessary for (38) to hold. Thus the above construction provides examples for situations when the lower factorization property is not satisfied while assumptions (a1) and (a2) hold true.

Example B.3. Let ρ and σ be the global Gibbs states of translation-invariant finite-range interactions Φ and Ψ , respectively. The local Gibbs state ρ_n^G for Φ has the density $e^{-H_n(\Phi)} / \text{Tr } e^{-H_n(\Phi)}$, where $H_n(\Phi)$ is the local Hamiltonian of Φ inside $[1, n]$; the local Gibbs state σ_n^G is defined similarly for Ψ . Since there is a constant $\lambda \geq 1$ (independent of n) such that $\lambda^{-1}\rho_n \leq \rho_n^G \leq \lambda\rho_n$ and $\lambda^{-1}\sigma_n \leq \sigma_n^G \leq \lambda\sigma_n$ (see [11, Lemma 2.1]), the function ψ is written as

$$\psi(s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{Tr } e^{-sH_n(\Phi)} e^{-(1-s)H_n(\Psi)} - sP(\Phi) - (1-s)P(\Psi), \quad s \in \mathbb{R},$$

where $P(\Phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{Tr} e^{-H_n(\Phi)}$, the pressure of Φ . We have $\psi(0) = \psi(1) = 0$, and the Golden-Thompson trace inequality yields

$$\psi(s) \geq \xi(s) := P(s\Phi + (1-s)\Psi) - sP(\Phi) - (1-s)P(\Psi), \quad s \in \mathbb{R}.$$

Here equality cannot hold in general as is immediately seen in the case of product states. Note that $\xi(s)$ corresponds to one of the candidates proposed in [22] to obtain the Quantum Hoeffding bound. By [11, Theorem 2.4 and Lemma 2.3] we notice that ξ is differentiable on \mathbb{R} and moreover

$$\begin{aligned} \xi'(0) &= \partial P_\Psi(\Phi - \Psi) - P(\Phi) + P(\Psi) \\ &= -\sigma(A_{\Phi-\Psi}) - P(\Phi) + s(\sigma) - \sigma(A_\Psi) \\ &= s(\sigma) - \sigma(A_\Phi) - P(\Phi) \\ &= -S_M(\sigma||\rho), \end{aligned}$$

where A_Ψ is the mean energy of Ψ and $s(\sigma)$ is the mean entropy of σ . Similarly $\xi'(1) = S_M(\rho||\sigma)$. Hence we have $\psi'(0) = -S_M(\sigma||\rho)$ and $\psi'(1) = S_M(\rho||\sigma)$ as long as ψ is differentiable at 0, 1. In particular, when $H_n(\Phi)$ and $H_n(\Psi)$ commute for all n , it is obvious that $\psi = \xi$. But this situation is essentially classical since for all n the densities of ρ_n and σ_n commute, too.

Appendix C

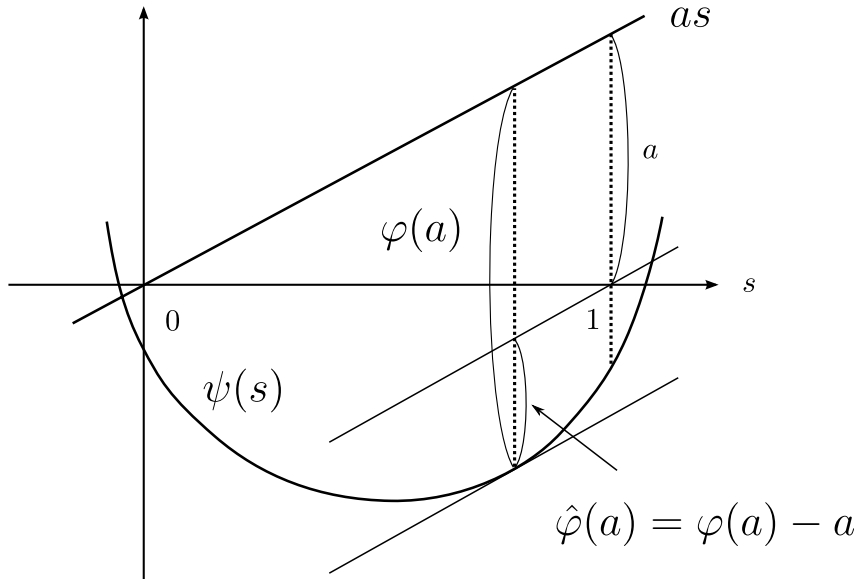


Figure 1: the definitions of φ and $\hat{\varphi}$ with a typical ψ

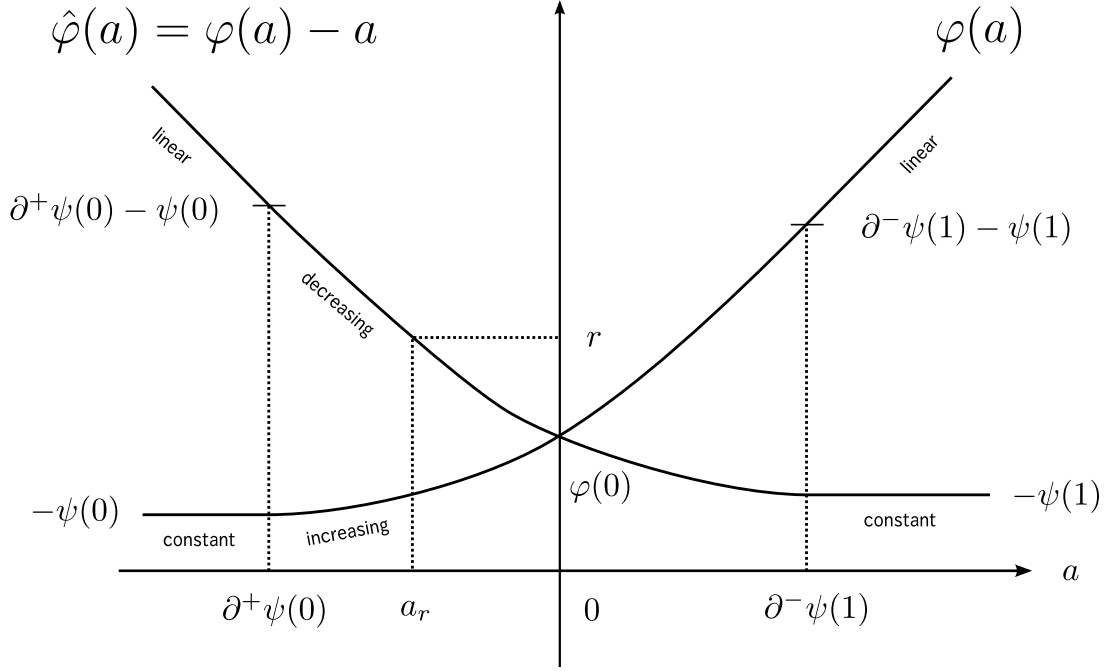


Figure 2: the graphs of φ and $\hat{\varphi}$ in a typical case

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